Strategic Complexity and the Core in Bargaining Games*

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Abstract

The aim of this paper is to make a non-cooperative foundation of core through complexity considerations in general bargaining games. We consider a coalitional bargaining with a set of deterministic protocols (order of proposers) which generally depend on histories in the entire bargaining. The set of protocols contains those discussed in much of the previous literature. Following Chatterjee and Sabourian (2000), we first show that when players have preferences that incorporate complexity of their strategies, every Nash equilibrium is stationary for general protocols. Second, we provide a necessary and sufficient condition that the core coincides with the set of allocations supported by pure-strategy stationary subgame perfect equilibria with threshold strategies. The condition requires the protocols to include rich patterns in the order of proposers.

Keywords: Strategic complexity, Core, Coalitional form, Bargaining protocol, Stationarity.

1 Introduction

This paper considers a general set of multi-player bargaining protocols (order of proposers), and provides a non-cooperative foundation of the core in a coalitional form game. In other

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words, we would like to characterize the core in terms of a solution concept in a non-cooperative bargaining model. Serrano (2005, page 250) suggests that introducing some kind of stationarity in the solution concept is one of the possible ways to accomplish this goal. We would like more, however, obtaining stationarity as a *conclusion* but not an assumption. A successful achievement is made by Chatterjee and Sabourian (2000), who introduce a notion of strategic complexity in a multi-player non-cooperative bargaining model, and prove that any Nash equilibrium is stationary if players incur small costs of implementing complex strategies. We show that their treatment applies for a generalized framework, and provide a new necessary and sufficient condition of the non-cooperative implementation of the core.

In this paper, we analyze a model of multi-player bargaining which is an extension of Rubinstein's (1982) well-known bargaining, in which two players propose alternatingly. There is no single model that is considered most appropriate as an extension of this protocol: Some works consider a protocol where a player who rejects an offer is the next proposer, and some consider players who are chosen as a proposer in a rotational manner. We introduce a general order of proposers, which is referred as a *protocol*, depending on the entire histories in the extensive form game. With this set of protocols, we investigate equilibria with complexity considerations, and consider conditions of the non-cooperative foundation of the core.

Our first objective is to prove stationarity from the assumption of complexity costs. In general, a stationary strategy is regarded as a strategy which prescribes the same action in any similar environment. Markov stationarity in terms of Maskin and Tirole (2001) requires a player to choose the same action after any payoff-irrelevant history. Let us consider subgames after nodes at which some player rejects an offer. In the bargaining with general protocols, the order of proposers in the subgames may differ, and this difference may yield different outcomes. Such subgames are payoff-relevant, and thus Markov strategies may play distinct actions in these subgames. To impose more restriction in the equilibrium concept, we may consider a stronger notion of stationarity which is called *payoff-oriented choice rule* (Okada and Winter (2002)). This notion requires a player's strategy to make the same response to all offers that allocate the same payoffs to the responder, whenever he is the last responder (i.e., an agreement is reached if he accepts the offer). We define strategic complexity as a partial ordering in the set of strategies to consider the general stationarity including the usual Markovian, and the Markovian combined with the payoff-oriented choice rule. In either notion, strategic complexity is defined so that any stationary strategy is minimally complex. Then we generalize Nash equilibrium with complexity costs (NEC) defined by Chatterjee and Sabourian (2000) to the general stationarity above. Our first main result is that every NEC is stationary in the generalized environment; in the general bargaining protocol, with general super-additive characteristic functions, and under the notions of general stationarity.

Next we provide a new characterization of the core, imposing a requirement called the *threshold rule*: A player always makes the same offer when he is a proposer, and when he is a responder, he accepts an offer in and only if it allocates to him larger than or equal to some threshold value, even when he is not the last responder.[1] This notion is stronger than the payoff-oriented choice rule since the threshold rule is imposed on all responders not only the last one. We show that the core coincides with the set of allocations supported by a pure-strategy stationary subgame perfect equilibrium (SSPE), where the threshold rule is assumed as a notion of stationarity. To obtain this result, we need a property of the bargaining protocols saying that it should include rich patterns in the bargaining. A easy sufficient condition of this rich patterns is that for any pair of players i, j there exists a history such that player i rejects some offer, and then player j is the proposer in the next round. This property is satisfied for example if every cyclic ordering of players generated by permutations appears in each subgame, but there are much more protocols that satisfy the above condition. We show that this "rich patterns" property of protocols is a necessary and sufficient condition of the non-cooperative foundation of the core for all super-additive characteristic functions.

Large literature investigates non-cooperative bargaining games where several players are involved. Selten (1981) considers a multi-person bargaining procedure which begins with an offer by a prespecified player, followed by responses of the other players. Then if some player rejects an offer, he immediately makes a counter-offer. Many papers analyze models with this kind of procedure. Chatterjee et al. (1993) investigate the multi-person bargaining with discounting and show that for coalitional form games with strict super-additivity, the stationary subgame perfect equilibrium payoffs must converge to an allocation in the core as the discount factor goes to unity if the first proposer in each coalition is appropriately selected. Moreover, they show that for strictly convex games, the limit equilibrium payoff profile is in the core and Lorenz dominates every other core allocation. Moldovanu and Winter (1995) consider the multi-person bargaining with no discounting and nontransferable utility. They introduce a stronger equilibrium concept requiring that a stationary subgame perfect equilibrium payoffs remains the same, independent of the choice of the first proposer in each coalition, and show that the set of pure-strategy equilibrium payoffs with this notion is equal to the core if the game is totally-balanced. Okada and Winter (2002) consider the model with no discounting, where the bargaining procedure is modified so that the proposer in the first round is determined as the proposer in every $(mK+1)$-th round for period $K \geq 2$. They show that the core coincides with the payoff set of pure-strategy stationary subgame perfect equilibria with the payoff-oriented choice rule.

[1] Okada (1992) refers to this property as "payoff-oriented response."

There are many other papers that establish a connection between the set of equilibrium payoffs and the core from various points of view. Perry and Reny (1994) investigate a multi-person bargaining model with continuous time, and show that for totally-balanced games, the set of stationary subgame perfect equilibrium payoffs is equal to the core. Evans (1997) considers a multi-person bargaining model where there is a competition for being a proposer before an offer is made in every round, and shows that the set of pure-strategy stationary subgame perfect equilibrium payoffs coincides with the core.

Strategic complexity, the assumption that players have preferences with which they avoid complex strategies if the payoffs remain the same, is considered in the context of repeated games in many papers including Rubinstein (1986), Abreu and Rubinstein (1988), Banks and Sundaram (1990), Binmore and Samuelson (1992), Piccione and Rubinstein (1993), Carmona (2006), Muto (2008), and many others. While all of the papers listed above consider two-player games, Chatterjee and Sabourian (2000) study multi-person games, but not repeated games. They investigate a bargaining game with discounting in which the characteristic function is the unanimous form, and show that players play stationary (i.e., minimally complex) strategies in any Nash equilibrium with such preferences with complexity considerations.[2] This result is significant because stationarity is obtained as a consequence of deduction, whereas most studies only presume it. A key to their consequence is that the bargaining game terminates if players come to an agreement, which makes a backward induction argument possible. Such intuition applies to other models with a similar property. Sabourian (2004) and Gale and Sabourian (2005) consider decentralized market models with many buyers and sellers, where a buyer-seller pair is supposed to exit the market when they complete bilateral trading. They show that, in each model, the competitive outcome is sustained in any equilibrium with complexity considerations.

The rest of the paper is organized as follows. In Section 2, we present the definition of coalitional form games and their properties, and a non-cooperative model of bargaining as well as a discussion and a definition of strategic complexity. In Section 3, we define the notion of Nash equilibrium with complexity costs following Chatterjee and Sabourian (2000), and prove that every NEC consists of stationary strategies. In Section 4, we demonstrate examples where inefficient stationary subgame perfect equilibria exist or not. In Section 5, we introduce the threshold response rule, and show that the set of pure-strategy SSPE payoff profile with this rule coincides with the core of the super-additive and totally-balanced coalitional form game if and only if the protocol have rich patterns. Section 6 concludes. Proofs are provided in Appendix.

[2] Binmore et al. (1998) study the two-person alternating-offer bargaining with complexity considerations in which evolutionary stability is adopted as an equilibrium concept.

2 Model

2.1 Coalitional Form Game

Let $N = \{1, \ldots, n\}$ ($n \geq 2$) be the set of players. A nonempty subset $S \subset N$ is a *coalition* of players. The set of all coalitions is denoted by $\mathcal{S}(N)$. A *characteristic function* $v : \mathcal{S}(N) \to \mathbb{R}$ is a real-valued function. The pair (N, v) is a *coalitional form game with transferable utility*. An allocation for coalition S is a $|S|$-dimensional vector $(x_i)_{i \in S}$ satisfying $\sum_{i \in S} x_i \leq v(S)$. Let X^S be the set of all allocations for S, and X_+^S be the subset of X^S where all components are nonnegative. An allocation in X_+^S is called *efficient* if $\sum_{i \in S} x_i = v(S)$.

The coalitional form game (N, v) is 0-*normalized* if $v(\{i\}) = 0$ for every player $i \in N$, and *super-additive* if $v(S \cup T) \geq v(S) + v(T)$ for any coalitions $S, T \in \mathcal{S}(N)$ with $S \cap T = \emptyset$. We assume these two properties throughout this paper. (N, v) has the *one-stage property* if $v(S) > 0$ implies that $v(T) = 0$ for every coalition T such that $S \cap T = \emptyset$. Note that for $n = 3$ the one-stage property is trivially implied by 0-normalizedness.

The *core* of (N, v) is a set of allocations for N defined by

$$\big\{(x_i)_{i \in N} \in X^N \mid \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \mathcal{S}(N)\big\}.$$

For a coalition $S \in \mathcal{S}(N)$, let $v|_S$ be the restriction of v on $\mathcal{S}(S)$, inducing the restricted game $(S, v|_S)$. The coalitional form game (N, v) is called *totally-balanced* if for any coalition $S \in \mathcal{S}(N)$ the restricted game $(S, v|_S)$ has a nonempty core.

2.2 Non-cooperative Bargaining Procedure

Bargaining Procedure

The non-cooperative bargaining procedure consists of a sequence of rounds $t = 1, 2, \ldots$. A player bargains about an allocation with the others, and leaves the procedure immediately after he makes an agreement in some coalition. For a given history in the game, a player still staying in the bargaining is referred to as an *active player*. Let us denote by N^t the set of active players in round t.

Each round t consists of three steps. First, some player i is selected as a proposer according to a deterministic (non-random) protocol.[3] This protocol, which is common knowledge among players, determines who to propose depending on the history which includes who

[3] We focus on the deterministic selection that may depend on the entire history of the extensive-form game, while stochastic recognition of proposers is also investigated in large literature including Baron and Ferejohn (1989), Okada (1996, 2011), Winter (1996), Yan (2002), and Compte and Jehiel (2010).

proposed and responded in the previous rounds. For simplicity, we assume that the choice of the proposer is independent of the allocation proposed in the previous round. Second, the proposer makes an offer (S, x^S) where $S \in \mathcal{S}(N^t)$ is a coalition including i, and x^S is an allocation in X^S. Third, each member in $S \setminus \{i\}$ sequentially responds either accepting or rejecting the offer. The order of responders does not affect our conclusion, and thus we do not specify it. Let A be the set of responses $\{\text{Yes}, \text{No}\}$. If a player $j \in S$ rejects the offer, then the bargaining immediately proceeds to round $t+1$ with the same set of active players $N^{t+1} = N^t$. If all members in S accept the offer, then they receive payoffs $(x_i)_{i \in S}$, and leave the bargaining procedure.[4] The remaining players in $N^{t+1} = N^t \setminus S$ bargain in round $t+1$.

The procedure continues until every member in N participates in a coalition. If the bargaining lasts infinitely many rounds, then any player who belongs to no coalition obtains zero payoffs.

Figure 1 illustrates an example of a partial game tree of this bargaining procedure with three players. Suppose that player 1 is the first proposer. There are three coalitions that contains 1; N, $\{1,2\}$, and $\{1,3\}$, to which 1 chooses to make an offer. If his offer is $(N, (x_1, x_2, x_3))$, then player 2 and 3 respond sequentially. If both responders accept the offer, then they agree with allocation (x_1, x_2, x_3). If one responder says "No," the next proposer makes a new offer in the second round. In this example, we assume that the second proposer is always player 2 no matter who rejects the offer, and no matter what offer is made. The first proposer 1 may propose to coalition $\{1,2\}$ as well. Then player 2 is the only responder to this offer. If he accepts it, the payoff profile $(x_1, x_2, 0)$ is received. (Recall that we assume 0-normalizedness, so that the remaining single-player coalition $\{3\}$ produces zero payoff only.) If he rejects the offer, then the next proposer is player 2 in this example.

History and the Protocol

We define the set of histories H in this bargaining procedure. Let $\mathcal{O}_i(S) = \{(T, x^T) \mid T \in \mathcal{S}(S), i \in T, x^T \in X^T\}$ be the set of offers that player $i \in S$ can make when the set of active players is S. A history consists of sequences of partial histories in each round. Suppose that S is the set of active players in round t given a history until round $t-1$. A *partial history* in round t when the proposer is player $i \in S$ is written as the sequence either $\big((T, x^T), \text{Yes}, \ldots, \text{Yes}\big)$ or $\big((T, x^T), \text{Yes}, \ldots, \text{Yes}, \text{No}\big)$ where $(T, x^T) \in \mathcal{O}_i(S)$ is an offer made by i, and the number of responses is nonnegative and smaller than $|T|$. The set of partial

[4] We assume that players are patient, and there is no discounting.

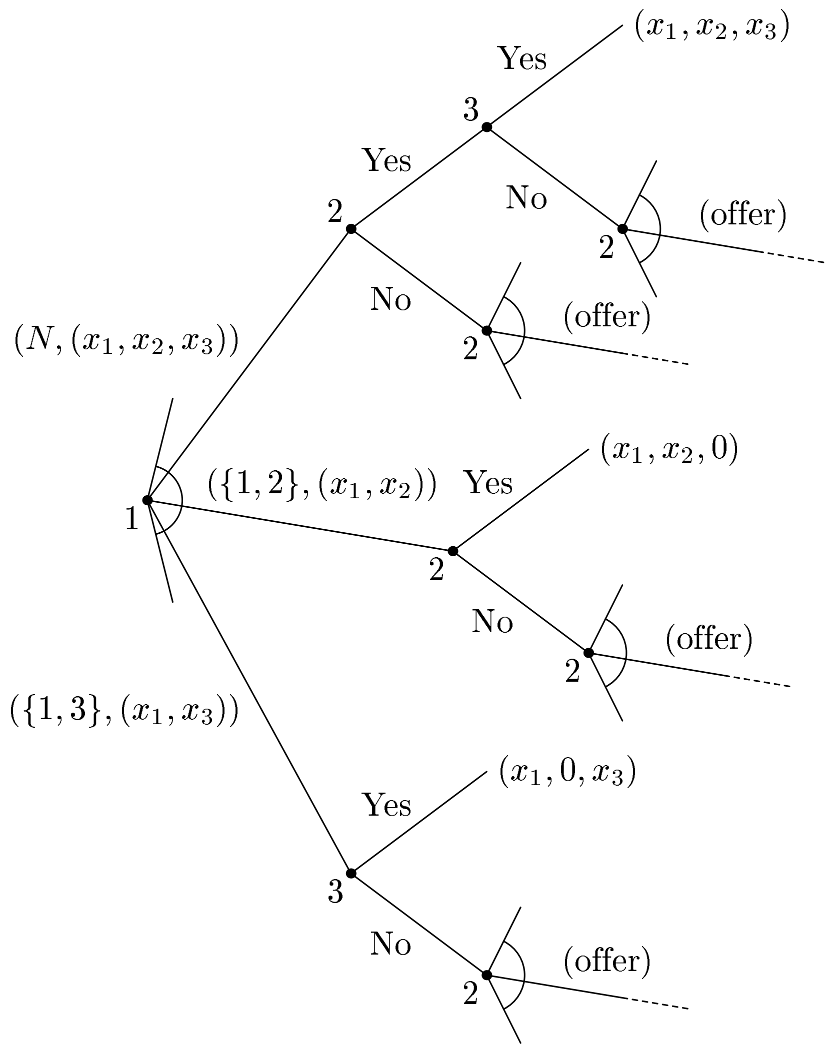

Figure 1: Partial game tree of the bargaining procedure with three players in which player 1 is the first proposer and 2 is the second.

histories in a round starting with player i's offer is denoted by $\mathcal{R}_i(S)$ and defined by

$$\begin{aligned}\mathcal{R}_i(S) = \{&\big((T, x^T), \overbrace{\text{Yes}, \ldots, \text{Yes}}^{k}\big) \mid (T, x^T) \in \mathcal{O}_i(S), 0 \leq k \leq |T| - 1\} \\ \cup \{&\big((T, x^T), \overbrace{\text{Yes}, \ldots, \text{Yes}}^{k}, \text{No}\big) \mid (T, x^T) \in \mathcal{O}_i(S), 0 \leq k \leq |T| - 2\}.\end{aligned}$$

We denote $\mathcal{R}(S) = \bigcup_{i \in S} \mathcal{R}_i(S)$.

Let $\mathcal{D}(S)$ be the subset of $\mathcal{R}(S)$ defined by

$$\mathcal{D}(S) = \{\big((T, x^T), \overbrace{\text{Yes}, \ldots, \text{Yes}}^{k}, \text{No}\big) \mid (T, x^T) \in \mathcal{O}_i(S), 0 \leq k \leq |T| - 2\}$$

which consists of partial histories after which a round ends with a rejection of an offer. Let $\mathcal{A}(S)$ be the set of partial histories after which a round ends with an agreement among all the members in a coalition, i.e.,

$$\mathcal{A}(S) = \{\big((T, x^T), \overbrace{\text{Yes}, \ldots, \text{Yes}}^{|T|-1}\big) \mid (T, x^T) \in \mathcal{O}_i(S)\}.$$

A round ends either with a rejection of an offer by some player or with an acceptance by all responders in the coalition. Therefore $\mathcal{D}(S) \cup \mathcal{A}(S)$ is the set of partial histories after which a round ends. All players in coalition T leave the bargaining process at the end of a partial history $\big((T, x^T), \text{Yes}, \ldots, \text{Yes}\big) \in \mathcal{A}(S)$ with an agreement, so that $S \setminus T$ is the set of active players in the next round.

Given a history $(h^1, \ldots, h^{t-1})$ until the end of round $t - 1$, a player is selected as a proposer at the beginning of round t if there are active players still continuing bargaining. Let us denote the player by $\varphi(h^1, \ldots, h^{t-1}) \in N$ who is selected as a proposer according to the protocol. The set of possible sequences of actions until a node between the beginning and the end of round t is recursively defined as

$$\begin{aligned}
&\tilde{H}^0 = \{\varnothing\} \text{ for the null history } \varnothing, \\
&\tilde{H}^1 = \mathcal{R}_{\varphi(\varnothing)}(N^1), \text{ where } \varphi(\varnothing) \text{ is the first proposer}, \\
&\tilde{H}^t = \big\{(h^1, \ldots, h^{t-1}, h^t)\big|(h^1, \ldots, h^{t-1}) \in \tilde{H}^{t-1}, h^{t-1} \in \mathcal{D}(N^{t-1}) \cup \mathcal{A}(N^{t-1}), \\
&\qquad N^t \neq \emptyset, h^t \in \mathcal{R}_{\varphi(h^1,\ldots,h^{t-1})}(N^t)\big\} \text{ for } t \geq 2,
\end{aligned}$$

where N^t is recursively expressed as

$$\begin{aligned}
&N^1 = N, \\
&N^t = \begin{cases} N^{t-1} & \text{if } h^{t-1} \in \mathcal{D}(N^{t-1}), \\ N^{t-1} \setminus S & \text{if } h^{t-1} \in \mathcal{A}(N^{t-1}), \text{ coalition } S \text{ is formed in round } t-1. \end{cases}
\end{aligned}$$

Let $N(h)$ be the set of active players after $h \in \tilde{H}^t$. Let

$$H^t = \tilde{H}^t \setminus \{(h^1, \ldots, h^t) \in \tilde{H}^t \mid \text{No active player remains after } h^t.\}$$

be the set of histories, after which the extensive-form game does not terminate. Let $H = \bigcup_{t\geq 0} H^t$ be the entire set of histories, and H_i be the set of histories after which it is player i's turn to move. Let $H_i^t = H_i \cap H^t$.

The protocol φ is a function from $\{(h^1, \ldots, h^t) \in H \mid h^t \in \mathcal{D}(N^t) \cup \mathcal{A}(N^t)\}$ to N. The proposer in the next round has to be determined if and only if the current node is a terminate node of a round, and there still exists a subsequent round played by remaining players. For simplicity of the subsequent analysis, we assume that φ does not depend on the allocation proposed in the current round. Despite this assumption, the set of possible protocols is still quite large, as it contains those dependent on the identity of proposers or rejectors, or the coalitions which was proposed to in the previous rounds.

Strategy

A pure strategy is defined in a standard manner. Player i's pure strategy σ_i is a function from H_i to $\bigcup_{S\in\mathcal{S}(N)} \mathcal{O}_i(S) \cup A$, the set of either offers or responses. We focus on pure strategies, which are henceforth referred to simply as strategies. Suppose that a history $h = (h^1, \ldots, h^t)$ is an element of H_i^t, after which it is player i's turn to either propose or respond. If round t ends immediately after h^t, then the next move is a proposal by player i in round $t + 1$. Thus $\sigma_i(h) \in \mathcal{O}_i(N^{t+1})$ where N^{t+1} is the set of active players in round $t+1$. If it is i's turn to respond, then $\sigma_i(h)$ is contained in $A = \{\text{Yes}, \text{No}\}$. Given strategies σ_i for every player i, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a strategy profile, and $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$ is the strategy profile excluding player i. The profile σ is sometimes written as (σ_i, σ_{-i}). A strategy profile σ determines a unique path in the extensive-form game, yielding a non-random bargaining outcome. Let $u_i(\sigma)$ be the payoff that player i receives with this outcome.

2.3 Strategic Complexity

In this subsection, we introduce a notion of strategic complexity, which is defined as a partial ordering in the set of strategies. This definition is a generalization of that in Chatterjee and Sabourian (2000) who consider the unanimous bargaining. Their notion of strategic complexity embodies an intuition that a player taking a "simple" strategy should behave in the same way in two subgames if they are isomorphic in some sense.

For ease of notation, we may identify a history $h \in H$ and the succeeding subgame associated with h, as our model is an extensive-form game with perfect information. A partition of histories mathematically represents which subgame is isomorphic with another. For a partition Π of H and a history $h \in H$, let $\Pi[h] \subset H$ be an element of Π that contains h. Let Π_i be the partition of H_i which is the restriction of Π. If a strategy σ_i is Π_i-measurable, i.e., $\tilde{h} \in \Pi_i[h]$ implies $\sigma_i(h) = \sigma_i(\tilde{h})$, then we call σ_i a Π-*stationary strategy*, as a Π-stationary strategy always assigns the same behavior after any histories in a partition element of Π_i. This is a generalization of stationarity defined by Maskin and Tirole (2001) in the context of general dynamic games with observable actions. Following their definition, we may define *payoff irrelevant partition* Π_i^{pi} of player i in our particular context of bargaining depending on the protocol φ. Then player i's strategy σ_i is a Markov strategy if and only if σ_i is Π^{pi}-stationary.

It is difficult to explicitly formulate the payoff irrelevant partition Π_i^{pi} for a general protocol φ. In the next two examples, we focus on specific classes of protocols which widely appear in the literature.[5] We describe how the bargaining procedure affects the set of Markov (or Π^{pi}-stationary) strategies.

Example 1 (Counter-Offer Protocol)**.** Consider a bargaining procedure where any proposal after the first round is a "counter-offer," which is an offer made by the responder who has rejected the previous offer. For a history $h = (h^1, \ldots, h^t) \in H$, suppose that $h^t \in \mathcal{D}(N^t)$. Then $\varphi(h)$ is the player who rejected the offer in the last response in h^t. (Recall that a round ends immediately after a rejection.) If $h^t \in \mathcal{A}(N^t)$, then the first proposer $\varphi(h)$ in the bargaining starting with the set of active players $S \in \mathcal{S}(N)$ is determined arbitrarily.

We can explicitly present the payoff irrelevant partition in this simple bargaining procedure. Two histories $h = (h^1, \ldots, h^t) \in H_i^t$ and $\tilde{h} = (\tilde{h}^1, \ldots, \tilde{h}^{t'}) \in H_i^{t'}$ belong to the same partition element in Π_i^{pi} if and only if (i) the set of active players are the same ($N(h) = N(\tilde{h})$), and (ii) either it is player i's turn to propose after $h, \tilde{h}$, or it is player i's turn to respond and $h^t = \tilde{h}^{t'}$. If player i plays a Markov strategy, then he should make the same proposal whenever it is his turn to propose, and make the same response to the same offer in every round.

Example 2 (Cyclic Protocol)**.** Consider a bargaining procedure with a cyclic order of proposers. For a history $h = (h^1, \ldots, h^t) \in H$, we define φ recursively as follows: Let $S = N(h)$. Count the number of rounds played with the set of active players S. If round $t+1$ is the first round of the bargaining with the set of active players S, then the proposer $\varphi(h)$ is $\min(S)$. If round $t+1$ is the second round, then $\varphi(h)$ is the player who has the second smallest index. In general, if round $t+1$ is the t'-th round with the set of active players S, then the proposer $\varphi(h)$ is the player with r-th smallest index where r is an integer with $1 \le r \le |S|$ such that $t' = m\,|S| + r$ for some integer m. Note that this protocol does not depend on who rejected the offer in the previous round.

[5] For example, Selten (1981), Chatterjee et al. (1993), Moldovanu and Winter (1995), and Horniaček (2008) consider the counter-offer protocol, while Chae and Yang (1988), Krishna and Serrano (1996), and Chatterjee and Sabourian (2000) analyze the cyclic protocol with a variety in the details.

Fix two histories $h = (h^1, \ldots, h^t) \in H_i^t$ and $\tilde{h} = (\tilde{h}^1, \ldots, \tilde{h}^{t'}) \in H_i^{t'}$, and suppose that the set of active players is still N after both histories h and $\tilde{h}$. If it is player i's turn to propose after $h, \tilde{h}$, then these histories belong to the same element of the payoff irrelevant partition. This holds only if $t - t'$ is a multiple of n. If it is player i's turn to respond after $h, \tilde{h}$, these histories belong to the same element in Π_i^{pi} if and only if (i) $h^t = \tilde{h}^{t'}$, and (ii) $t - t'$ is a multiple of n (i.e., the same player proposes in rounds t, t'). Note that the second condition does not appear in Example 1. Unlike the previous example, a player playing a Markov strategy can make different responses to the same offer made in distinct rounds. This is because two histories are payoff relevant if there come different proposers in subsequent rounds.

It is well-known that the set of (non-stationary) subgame perfect equilibrium payoffs tends to be very broad, covering the set of all efficient allocations in many cases, if three or more players are involved in the bargaining. For this reason, literature usually makes a Markovian assumption about strategies to refine the set of equilibrium outcomes. In our case with general protocols, however, the Markovian property may be of little use. For instance, let us consider the cyclic protocol with period K. Two histories h, $\tilde{h}$ belong to the same partition in Π^{pi} if the set of active players after h coincides with that after $\tilde{h}$, $h^t = \tilde{h}^{t'}$, and $t - t'$ is a multiple of K. Therefore the greater K is, the smaller each partition element of Π^{pi} can be. The payoff irrelevant partition is finer in such a case than in the well-known protocols in the literature, and thus the Markovian property may impose little restriction.

This observation motivates us to introduce a stronger notion of stationarity to find the set of bargaining outcomes that seems reasonable. Okada and Winter (2002) introduce a concept called *payoff-oriented choice rule*. This rule requires that, for a set of active players S, a player $i \in S$ who responds in the last (i.e., members agree with the offer if he accepts it) should set a reservation price p_i^S, accepting an offer if and only if it allocates to him a payoff no smaller than p_i^S. This requirement is independent of the Markovian property since p_i^S is independent of the history, e.g., the identity of the proposer in the round, the allocation for other players in the coalition, and the index of the round.

Let Π^{po} be the partition corresponding to the above requirement of stationarity in addition to the payoff irrelevance. Formally, two histories $h = (h^1, \ldots, h^{t-1}, h^t) \in H_i^t$ and $\tilde{h} = (\tilde{h}^1, \ldots, \tilde{h}^{t'-1}, \tilde{h}^{t'}) \in H_i^{t'}$ belong to the same element in Π^{po} if and only if (i) the sets of active players after h and $\tilde{h}$ coincide, (let $S = N(h) = N(\tilde{h})$), and (ii) either of the following statements holds: (a) h and $\tilde{h}$ belong to the same element in the payoff irrelevant partition Π^{pi}, or (b) i is the last responder after $h, \tilde{h}$, and the offers $(T, (x_j)_{j \in T})$, $(T', (x'_j)_{j \in T'})$ in h^t and $\tilde{h}^{t'}$, respectively, both allocate to i the same amount $x_i = x'_i$. Note that Π^{po} is strictly coarser than Π^{pi}. We may also consider the intermediate notion of stationarity. Let $\mathcal{P}$ be the set of partitions of H no finer than Π^{pi} and no coarser than Π^{po}. Let $\mathcal{P}_i$ be the set of partitions of H_i which is the restriction of partitions in $\mathcal{P}$.

Following Chatterjee and Sabourian (2000), we adopt a notion of complexity that is sufficient for proving the results.

Definition 1. For a partition of histories $\Pi_i \in \mathcal{P}_i$ for a player i, we define a transitive relation denoted by $\succsim^{\Pi_i}$ to be generated by the following relations: For two strategies σ_i, σ_i' of player i, $\sigma_i \succsim^{\Pi_i} \sigma_i'$ if $\sigma_i = \sigma_i'$, or there is some history $\tilde{h} \in H_i$ such that

$$\begin{aligned}
&\sigma_i(h) = \sigma_i'(h) && \text{for all } h \notin \Pi_i[\tilde{h}],\\
&\sigma_i(\tilde{h}) \neq \sigma_i(h) && \text{for some } h \in \Pi_i[\tilde{h}], \text{ and}\\
&\sigma_i' \text{ is constant on } \Pi_i[\tilde{h}].
\end{aligned}$$

If $\sigma_i \succsim^{\Pi_i} \sigma_i'$ and $\sigma_i \neq \sigma_i'$, we say that σ_i is more Π_i-complex than σ_i' (or equivalently, σ_i' is Π_i-simpler than σ_i).

In Definition 1, σ_i is more Π_i-complex than σ_i' if two strategies are identical except that σ_i' assigns the same action in a partition element $\Pi_i[\tilde{h}]$ while σ_i does not. This definition is an extension of the notion of Chatterjee and Sabourian (2000) who consider the n-player unanimity bargaining. Sabourian (2004), and Gale and Sabourian (2005) consider a decentralized market model where a similar notion of complexity consideration is adopted.[6]

For a partition $\Pi \in \mathcal{P}$, a strategy σ_i is a minimal element with respect to the partial relation $\succsim^{\Pi_i}$ if and only if σ_i is Π-stationary, playing the same action after any history in a partition element in Π_i. Similarly, for a history $h \in H_i$, we say that σ_i is *stationary on* $\Pi_i[h]$ if σ_i plays the same action after every history in $\Pi_i[h]$. Note that σ_i is Π-stationary if and only if σ_i is stationary on $\Pi_i[h]$ for all $h \in \Pi_i$.

3 Nash Equilibrium with Complexity Costs

In this section, we define a notion of a Nash equilibrium with complexity costs (NEC) for a partition of histories $\Pi \in \mathcal{P}$, and show that any pure-strategy NEC is Π-stationary.

We consider the case in which players incur costs of implementing a strategy which are infinitesimally small compared to the payoffs obtained in the bargaining. A player is supposed to choose the simpler strategy of the two if they give the player equal payoffs. We fix a partition $\Pi \in \mathcal{P}$.

[6] Lee and Sabourian (2007) adopts a weaker measure of strategic complexity in a negotiation game.

Definition 2. For partition $\Pi \in \mathcal{P}$, a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a Π-*Nash equilibrium with complexity costs* (Π-*NEC*) if for any player i, (i) $u_i(\sigma) \geq u_i(\sigma'_i, \sigma_{-i})$ for all strategy σ'_i of player i, and (ii) if $u_i(\sigma) = u_i(\sigma'_i, \sigma_{-i})$ then σ'_i is not Π_i-simpler than σ_i for any σ'_i. We say that a Π-NEC is Π-stationary if every player plays a Π-stationary strategy in the equilibrium.

Clearly, if Π is finer than Π', then the set of Π-NEC contains the set of Π'-NEC.

The following proposition shows that a Π-NEC is always Π-stationary. As Chatterjee and Sabourian (2000) indicate, their proof works well also in our generalized setup.

Proposition 1. *For a partition $\Pi \in \mathcal{P}$, if a (pure) strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a Π-NEC, then σ_i is Π-stationary for all i.*

The proof is given in Appendix. This proposition shows that we may obtain stationarity as a consequence of strategic interaction, not just an ad hoc assumption. Furthermore, this conclusion applies to quite a general setup; general protocol φ, general characteristic function v, and a general stationarity notion expressed by a partition $\Pi \in \mathcal{P}$.

4 Stationary Subgame Perfect Equilibria — Examples

We showed in the previous section that for a partition $\Pi \in \mathcal{P}$ of histories, any Π-NEC is Π-stationary. In subsequent sections, we will investigate properties of the set of subgame perfect equilibrium which consists of Π-stationary strategies.

Definition 3. For a partition $\Pi \in \mathcal{P}$, a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a Π-*stationary subgame perfect equilibrium* (Π-*SSPE*) if σ is a subgame perfect equilibrium and each σ_i is a Π-stationary strategy.

First, we present a few examples of stationary subgame perfect equilibria in coalitional bargaining games. First, let us consider the unanimity bargaining game discussed by Chatterjee and Sabourian (2000).

Example 3. Let $v(S)$ be the unanimity characteristic function, namely,

$$v(S) = \begin{cases} 1 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

Then any efficient allocation $(x_i)_{i \in N} \in X_+^N$ belongs to the core of (N, v).

Players become a proposer according to the cyclic protocol defined in Example 2. Since the unanimity game has the one-stage property, the bargaining ends immediately when a coalition is formed. The proposer in round t is player i whenever $t = i + mn$ for some integer m. For $\Pi \in \mathcal{P}$ and any allocation $x = (x_i)_{i \in N}$, the following strategy profile supports x as a Π-SSPE:

- Each player i offers x to the grand coalition N when he is to make an offer.
- Each player i accepts offer $y = (y_j)_{j \in S}$ if and only if $y_i \geq x_i$, regardless of who is the proposer.

Therefore, the set of Π-SSPE payoffs coincides with the core for any Π in this example.

Chatterjee and Sabourian (2000) investigate the n-player unanimity bargaining model with this cyclic protocol, but with a discount factor $\delta < 1$.[7] They mainly argue stationarity with partition Π^{pi}, whereas they introduce another stationarity in the last section of their paper, which requires that a player should respond the same to the same allocation in every round regardless of who to propose. This example shows either specification of stationarity leads to the same conclusion, as they indicate.

In the above example, the core equals the set of Π-SSPE allocations for any Π. However, this coincidence does not hold true for general characteristic functions, as demonstrated in the next three examples.

Example 4. Let $N = \{1, 2, 3\}$, and $v(N) = 1$, $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0.6$. Let φ be the cyclic protocol considered in Example 2. We have an inefficient allocation which is supported by a Π^{pi}-SSPE:

- Player 1 proposes $(x_1, x_3) = (0.6, 0)$ to coalition $\{1, 3\}$, and accepts any offer made by player 2, or offer x with $x_1 \geq 0.6$.
- Player 2 proposes $(x_1, x_2) = (0, 0.6)$ to coalition $\{1, 2\}$, and accepts any offer made by player 3, or offer x with $x_2 \geq 0.6$.
- Player 3 proposes $(x_2, x_3) = (0, 0.6)$ to coalition $\{2, 3\}$, and accepts any offer made by player 1, or offer x with $x_3 \geq 0.6$.

This strategy profile is indeed Π^{pi}-stationary, as their responses depend only on the pair of the proposer and the offer.

The strategies in the above example are, however, not Π^{po}-stationary, since they adopt different threshold values according to the identity of the proposer. One might expect from this observation that Π^{po}-stationarity would be sufficient to obtain the coincidence between

[7] The set of SSPE payoffs is singleton with discount factor δ smaller than one in this example.

core and the set of SSPE payoffs. Then next examples show that Π^{po}-stationarity is still insufficient.

Example 5. Let $N = \{1, 2, 3\}$, and $v(N) = 1$, $v(\{1, 2\}) = v(\{1, 3\}) = 0.8$, $v(\{2, 3\}) = 0$. Let φ be the counter-offer protocol considered in Example 1. The following strategy profile supports an inefficient equilibrium.

- Player 1 proposes $(x_1, x_2) = (0.4, 0.4)$ to coalition $\{1, 2\}$, and accepts any offer x with $x_1 \geq 0.4$.
- Player 2 proposes $(x_1, x_2) = (0.4, 0.4)$ to coalition $\{1, 2\}$, and accepts any offer x with $x_2 \geq 0.4$.
- Player 3 proposes $(x_1, x_3) = (0.4, 0.4)$ to coalition $\{1, 3\}$, and accepts any offer x with $x_3 \geq 0.4$.

Example 6. Let $N = \{1, 2, 3\}$, and $v(N) = 1$, $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0.6$. Consider the cyclic protocol in Example 4. We have an inefficient allocation supported by a Π^{po}-SSPE:

- Player 1 proposes $(x_1, x_2) = (0.3, 0.3)$ to coalition $\{1, 2\}$, and accepts any offer x with $x_1 \geq 0.3$.
- Player 2 proposes $(x_1, x_2, x_3) = (0.3, 0.3, 0.4)$ to coalition N, and accepts any offer x with $x_2 \geq 0.3$.
- Player 3 proposes $(x_1, x_2, x_3) = (0.3, 0.3, 0.4)$ to coalition N, and accepts any offer x with $x_3 \geq 0.4$.

Next we introduce a non-standard class of protocols called an "anti-counteroffer protocol," in which no player can be a proposer after he rejects an offer.

Example 7. Let $N = \{1, 2, 3\}$, and $v(N) = 1$, $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0.6$. Consider an anti-counteroffer protocol. We have the following Π^{po}-SSPE that supports an outcome that does not belong to the core.

- Player 1 proposes $(x_1, x_2, x_3) = (1, 0, 0)$ to coalition N, and accepts any offer.
- Player 2 proposes $(x_1, x_2, x_3) = (0, 1, 0)$ to coalition N, and accepts any offer.
- Player 3 proposes $(x_1, x_2, x_3) = (0, 0, 1)$ to coalition N, and accepts any offer.

These observations lead us to introduce a property about the protocols, which will be discussed in Section 5.

5 Core and Stationary Subgame Perfect Equilibria

In the previous section, we observed that a coalitional bargaining game may have inefficient equilibria even if players play simple strategies. In this section, we provide conditions under which the core of (N, v) is equal to the set of pure-strategy SSPE outcomes.

Let us consider a stationarity condition called the *threshold rule*:[8] For a set of active players S, a player $i \in S$ makes the same offer when he is a proposer, and when i is a responder, he sets a reservation price p_i^S, and accepts an offer if and only if it allocates to him a payoff no smaller than p_i^S. Recall that this requirement is stronger than the payoff-oriented response rule by Okada and Winter (2002) in that the threshold rule does require threshold responses for all responders, not only for the last one. Let us denote Π^{tr} by the partition of the set of histories H defined by this rule. Formally, two histories $h = (h^1, \ldots, h^{t-1}, h^t) \in H_i^t$ and $\tilde{h} = (\tilde{h}^1, \ldots, \tilde{h}^{t'-1}, \tilde{h}^{t'}) \in H_i^{t'}$ belong to the same element in Π^{tr} if and only if (i) the sets of active players after h and $\tilde{h}$ coincide, and (ii) either of the following statements holds: (a) it is the same player's turn to propose after h and $\tilde{h}$, or (b) the offers $(T, (x_j)_{j \in T})$, $(T', (x'_j)_{j \in T'})$ in h^t and $\tilde{h}^{t'}$, respectively, both allocate to i the same amount $x_i = x'_i$. We hereafter focus on this partition Π^{tr}.

This requirement of taking threshold strategies might seem very strong. Nonetheless, we can show a lemma claiming that every core allocation is supported by a Π^{tr}-stationary subgame perfect equilibrium whenever (N, v) is totally-balanced.

Lemma 1. *Suppose that $x = (x_i)_{i \in N} \in X_+^N$ be an allocation that belongs to the core. Then x is supported by a pure-strategy Π^{tr}-stationary subgame perfect equilibrium if (N, v) is totally-balanced.*

In the proof, we just argue that the usual threshold strategy profile that always offers the same core allocation constitutes an equilibrium.

The converse of Lemma 1 is not true in general, as we demonstrated in Section 4. To identify a condition that no such inefficient equilibria exist, we define a property of protocol φ which can be interpreted to mean that the protocol involves sufficiently rich patterns. This property essentially says that an event "j proposes immediately after a rejection by i" happens in some history for most pairs of players (i, j).

Definition 4 (Rich patterns)**.** For coalition $S \in \mathcal{S}(N)$, let $M(S) \subset S^2$ be the set of pairs of players $(i, j) \in S^2$ (possibly $i = j$) that satisfy the following condition: There exists a history $h = (h^1, \ldots, h^t) \in H^t$ such that, in round t, the set of active players is S, a proposer

[8] A similar notion is considered by Okada (1992).

proposes to the grand coalition S, player i rejects the offer, and

$$\varphi(h) = j,$$

that is, player j is the proposer in the next round $t+1$. Protocol φ is said to have *rich patterns* if $(i,j) \in M(S)$ for all $(i,j) \in S^2$ with $i \neq j$, and $(i,i) \in M(S)$ for some $i \in S$.

A player generally tends to obtain larger payoffs when he is a proposer than when he is a responder because a proposer is able to choose any allocation he likes. Therefore a player has a larger bargaining power when he will be the next proposer if someone rejects the offer. The Π^{tr}-stationarity combined with this property requires that players should play identically in every round even though they may have different bargaining powers in different subgames.

Let us consider the case with three players. The cyclic protocol in Example 6 does not have rich patterns, as can be seen as follows. Suppose that player 1 will be the proposer in round $t+1$. Then the proposer in round t must be player 3, implying that 3 cannot be a rejector in round t. Therefore player 1 cannot be the next proposer after a round in which player 3 rejects an offer. The next example presents a protocol which has rich patterns.

Example 8. Let $N = \{1,2,3\}$. Consider the following periodic protocol with period 6: Suppose that the identity of the proposer depends only on the index of the round. Let $\varphi^t \in N$ be the proposer in round t, and

$$\varphi^1 = 1, \varphi^2 = 2, \varphi^3 = 3, \varphi^4 = 1, \varphi^5 = 3, \varphi^6 = 2, \ldots.$$

This protocol has rich patterns. Indeed, for example, player 1 is the proposer in round 4 after 1 or 2 rejects an offer in round 3, and is the proposer in round 7 after 3 rejects an offer in round 6.

Now let us revisit the strategy profile defined in Example 6:

- Player 1 proposes $(x_1, x_2) = (0.3, 0.3)$ to coalition $\{1,2\}$, and accepts any offer x with $x_1 \geq 0.3$.
- Player 2 proposes $(x_1, x_2, x_3) = (0.3, 0.3, 0.4)$ to coalition N, and accepts any offer x with $x_2 \geq 0.3$.
- Player 3 proposes $(x_1, x_2, x_3) = (0.3, 0.3, 0.4)$ to coalition N, and accepts any offer x with $x_3 \geq 0.4$.

In contrast to Example 6, the above profile is not a subgame perfect equilibrium. To see why, focus on the subgame after player 2 at round 6 deviates from the equilibrium path to proposing $x' = (x'_1, x'_2, x'_3) = (0.3, 0.6, 0.1)$ to coalition N. Then player 3 should accept this offer since if he rejected offer x', the next proposer 1 would follow the above strategy to exclude 3 from the coalition. Therefore, the above strategy of player 3 is not optimal in this subgame.

Note that the above argument relies on the fact that there are nodes at which player 1 proposes after player 3 rejects an offer. Such histories do not appear in the protocol in Example 6. The lack of a pattern in the protocol causes inefficiency in some equilibria. We will see that this insight extends to general cases.

We say that the core of (N, v) is *perfectly implemented* with threshold strategies if in any subgame starting at a proposal node at which all players are still active, the core is equal to the set of all continuation payoffs supported by pure-strategy Π^{tr}-SSPE. Our second main result is the following:

Proposition 2. *Let $n \geq 3$. The core of (N, v) is perfectly implemented with threshold strategies for any super-additive, 0-normalized, and totally-balanced characteristic function v if and only if the bargaining protocol φ has rich patterns.*

This result provides a non-cooperative implementation of the core in a setting with general protocols, and specifies a necessary and sufficient condition of the protocol for the implementation. The proof will directly follow from the next two lammata.

First, we show that if a protocol does not have rich patterns, then an inefficient SSPE for some characteristic function v which is super-additive, 0-normalized, and totally-balanced.

Lemma 2. *Suppose that $n \geq 3$, and protocol φ does not have rich patterns. Then there exists a super-additive, 0-normalized, and totally-balanced characteristic function v such that (N, v) is not perfectly implemented with threshold strategies.*

In the proof we construct a characteristic function similar to those defined in Example 6 and 7.

Next we show that if the protocol has rich patterns, any Π^{tr}-SSPE outcome is a core allocation.

Lemma 3. *Suppose that the protocol has rich patterns. Then any allocation supported by a pure-strategy Π^{tr}-stationary subgame perfect equilibrium is contained in the core of (N, v).*

Note that this lemma does not require totally-balancedness of v. If v is not totally-balanced, Lemma 3 implies that there is no pure-strategy Π^{tr}-SSPE.

6 Conclusion

In this paper, we have shown that stationarity in strategies is ensured in any Nash equilibrium with complexity costs, where we impose the stationarity as either the usual Markovian or the payoff-oriented response rule with Markovian in a bargaining with general protocols. The proof basically follows Chatterjee and Sabourian (2000) who consider the unanimity bargaining and the Markovian stationarity. Next we focus on the threshold rule of a strategy: A player always proposes the same offer, and has a threshold value in his responses which depends only on the set of active players, and on the payoff allocated to him in an offer. We provided a non-cooperative foundation of the core, showing that the core of a coalitional form game, for all super-additive and totally-balanced characteristic functions, is characterized as the set of allocations supported by a pure-strategy subgame perfect equilibrium with threshold strategies if and only if the protocol has rich patterns so that any player can be a proposer in the next round after another player rejects an offer, and there is at least one player who will be the next proposer after he rejects some offer.

Appendix

Proof of Proposition 1. If a player reaches no agreement on the equilibrium path, then he plays a Π-stationary strategy since no agreement is the worst outcome for him. We assume that an agreement is reached on the path played by σ.

Let us focus on round t on the path played by σ. We will show that for history $h \in H_i^t$, σ_i is stationary on $\Pi_i[h]$. It suffices to show that $\sigma_i(h) = \sigma_i(h')$ if both h and h' are histories on the path, since off-the-path actions do not affect payoffs.

Step 1: We show that if $h = (h^1, \ldots, h^t)$ is a history after which player i is the last responder, σ_i is stationary on $\Pi_i[h]$. Suppose that i is the last responder in round t on the path. Let $\big(S, x\big)$ be the offer in round t where $S \in \mathcal{S}(N)$ is a coalition and $x = (x_i)_{i \in S} \in X_+^S$ is an allocation. Player i either accepts or rejects this offer.

First suppose that i accepts this offer after h, obtaining his allocation x_i. Assume on the contrary that σ_i is not stationary on $\Pi_i[h]$. Then there exists a round $t' < t$ and a history $h' = (h^1, \ldots, h^{t'}) \in \Pi_i[h]$ after which i rejects the offer $\big(S', x'\big)$. By the definition of Π, we must have $x_i = x'_i$. Now consider the following strategy σ'_i which is Π_i-simpler than or equal to σ_i:

$$\sigma'_i(\tilde{h}) = \begin{cases} \sigma_i(h) \ (= \text{Yes}) & \text{if } \tilde{h} \in \Pi_i[h] \\ \sigma_i(\tilde{h}) & \text{otherwise.} \end{cases}$$

By the definition of $\mathcal{P}$ player i is the last responder also after all histories in $\Pi_i[h]$, and will obtain the same payoff if he accepts the offer. Therefore (σ_i', σ_{-i}) yields the same payoff for i. This is a profitable deviation by player i. By the definition of Π-NEC, σ_i is stationary on $\Pi_i[h]$ if i accepts the offer after $h \in H_i$.

Second suppose that player i rejects the offer after h. Assume on the contrary that σ_i is not stationary on $\Pi_i[h]$. Then there is a history h after which i accepts the offer on the path. Then the proof in the previous paragraph applies.

Step 2: Now let us start the backward induction. Let $h = (h^1, \ldots, h^t)$ be the entire history played by σ after which all players form coalitions. For integer $l \geq 0$, we denote by $h(l)$ the history given by erasing the last l actions. For the last responder i, Step 1 showed that σ_i is stationary on $\Pi_i[h(1)]$.

Suppose that player j is the second last responder, and the current offer is made to S where $|S| \geq 3$. Let $h(2)^t = ((S,x), \overbrace{\text{Yes}, \ldots, \text{Yes}}^{|S|-3})$. Suppose that there is an earlier round $t' < t$ such that $\tilde{h} = (h^1, \ldots, h^{t-1}, \tilde{h}^{t'}) \in \Pi_j[\tilde{h}]$ where $\tilde{h}^{t'}$ is the partial history in round t' which is a part of $h^{t'}$. By the definition of the payoff irrelevant partition, player j is the $(|S|-2)$-th responder also in round t'. Since player i would accept the offer if j accepted it after $\tilde{h}'$, j must have rejected the offer after $\tilde{h}'$. Then replacing σ_j by σ_j' which always assigns Yes in $\Pi_j[h(2)]$ strictly reduces the Π_j-complexity while retaining his payoff the same. Hence σ_j is stationary on $\Pi_j[\tilde{h}]$.

Step 3: To continue the backward induction, let k be the player who plays after $h(l)$ for $l \geq 3$. Suppose that σ is stationary on $\Pi[h(m)]$ for all $m < l$. If there is a history $\tilde{h} \in \Pi_k[h(l)]$ such that $\tilde{h} = h(L)$ for some $L > l$, and $\sigma_k(h(l)) \neq \sigma_k(\tilde{h})$. If k is the last responder, Step 1 showed stationarity on $\Pi[h(l)]$. If k is not the last responder, $h(l)$ and $\tilde{h}$ belong to the same payoff irrelevant partition. Therefore, replacing σ_k by σ_k' which always assigns $\sigma_k(h(l))$ in $\Pi_k[h(l)]$ strictly reduces the Π_k-complexity while retaining his payoff the same. This backward argument continues until there is a history $h(l)$ which is contained in $\Pi[\varnothing]$. shows that σ_k is stationary on $\Pi_k[h(l)]$ for all l.

Hence we have shown that for every player i σ_i is stationary on $\Pi_i[h]$ if history h is on the path. If all histories in $\Pi_i[h]$ appear off the equilibrium path for some history $h \in H_i$, then σ_i must be stationary on $\Pi_i[h]$ by the definition of NEC. Hence σ_i is a Π-stationary strategy for all $i \in N$, as desired. □

Proof of Lemma 1. Let σ_i be i's pure strategy. For each set of active players $S \in \mathcal{S}(N)$, suppose that every player $i \in S$ playing σ_i makes an offer $(x_j^S)_{j \in S} \in X_+^S$ to the grand coalition S whenever he is a proposer, accepts any offer allocating to him a payoff larger than or equal to x_i, and rejects any offer smaller than x_i when he is a responder. Since (N, v) is totally-balanced, $(x_j^S)_{j \in S}$ is a core allocation in the restricted game $(S, v|_S)$. Then we can show that $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a Π^{po}-SSPE.

Suppose that player i could earn a larger payoff than x_i by deviating to another strategy in a round with the set of active players S. Since the other players always offer x_i to i, the only chance of deviation must be making another offer $(x'_j)_{j \in T} \in X_+^T$ to some coalition $T \in \mathcal{S}(S)$ when it is his turn to propose. To ensure that the offer $\big(T, (x'_j)_{j \in T}\big)$ is accepted by all players in $T \setminus \{i\}$, we must have $x'_j \geq x_j$ for all $j \in T \setminus \{i\}$ by the definition of σ_j. Summing up in all $j \in T$, we have $\sum_{j \in T} x_j \leq \sum_{j \in T} x'_j \leq v(T)$. Since this deviation is profitable for i, we have $x_i < x'_i$, which implies that $\sum_{j \in T} x_j < v(T)$. This inequality contradicts to the premise that x is a core allocation. □

Proof of Lemma 2. Since protocol φ does not have rich patterns, at least one of the following two cases must hold. Case 1; there are two players $i, j \in N$ with $i \neq j$ such that in any history, player j cannot be the proposer after player i rejects an offer, or Case 2; any player $i \in N$ cannot be the proposer after i himself rejects an offer. In each case, we will construct a counter-example of a super-additive characteristic function under which there exists a SSPE with threshold strategies that yields an outcome outside of the core.

Case 1: Let v be a characteristic function with $v(N) = 1$, $v(N \setminus \{i\}) = 0.6$, and $v(S) = 0$ for the other coalitions S. Then v is obviously super-additive, 0-normalized, and totally-balanced. (Since $n \geq 3$, $N \setminus \{i\}$ contains more than one players.) Let us consider a threshold strategy profile described as follows:

- Player j proposes $\big(N \setminus \{i\}, (x_k)_{k \in N \setminus \{i\}}\big)$ with $x_k = 0.6/(n-1)$ for all $k \neq i$.
- The other players $k \in N \setminus \{j\}$ propose $\big(N, (x_l)_{l \in N}\big)$ where $x_i = 0.4$ and $x_k = 0.6/(n-1)$ for all $k \neq i$.
- Player i accepts an offer $\big(S, (x_l)_{l \in S}\big)$ if and only if $x_i \geq 0.4$.
- The other players $k \in N \setminus \{i\}$ accept an offer $\big(S, (x_l)_{l \in S}\big)$ if and only if $x_k \geq 0.6/(n-1)$.

This strategy profile yield an inefficient outcome with $x_i = 0$ when j is the proposer.

Now we show that this strategy profile is indeed an equilibrium. As long as all responders follow the above threshold strategies, no proposer has an incentive to deviate. Since allocations offered to player $k \in N \setminus \{i\}$ are the same at every opportunity, there is no profitable deviation when he is a responder. Suppose that a proposer offers to player i an allocation smaller than 0.4. If i rejects this offer, then the next proposer will be some player k $(\neq j)$, who offers 0.4 to i. Therefore player i optimally follows the prescribed action, rejecting the first offer. Hence the above strategy profile is a subgame perfect equilibrium.

Case 2: Let v be a characteristic function with $v(N) = 1$, $v(N \setminus \{i\}) = 0.6$ for all $i \in N$, and $v(S) = 0$ for the other coalitions S. Then v is obviously super-additive, 0-normalized, and totally-balanced. Let us consider a threshold strategy profile described as follows:

- Every player i proposes $\big(N, (x_j)_{j\in N}\big)$ with $x_i = 1$, and $x_j = 0$ for all $j \neq i$.
- Every player i accepts any offer.

This strategy profile yields an outcome that does not belong to the core.

Now we show that this strategy profile is indeed an equilibrium. No proposer has an incentive to deviate because the proposer obtains a maximum feasible payoff. Suppose that a player i rejects an offer. Then the next proposer is another player by the assumption. No matter how many times player i rejects offers, i cannot be a proposer as long as the other players follow the prescribed strategy profile. Therefore the above strategy profile is a subgame perfect equilibrium. □

Proof of Lemma 3. The proof proceeds as an induction argument with respect to n. If all offers are rejected in the entire history, every player will obtain zero payoffs. In such a case some player must deviate to earn a positive payoff. Therefore we can assume that there is an accepted offer.

Step 1: Assume that (N, v) has the one-stage property, which is always satisfied if $n \leq 3$. Denote player i's offer by $(S^i, (x^i_j)_{j\in S^i})$, and the vector of threshold prices by $p \in \mathbb{R}^n_+$. Let $x^i_k = 0$ for $k \notin S^i$. Let us denote $S = \{j \in N \,|\, p_j > 0\}$.

Step 1-1: We show that $S \subset S^i$ if i's offer is accepted. Suppose that $j \notin S^i$ and $p_j > 0$. Since the protocol has rich patterns there exists a history in which some player k $(\neq j)$ makes an offer, j rejects it, and the next proposer is i. Then in the subgame after k offered to j a small surplus ε with $p_j > \varepsilon > 0 = x^i_j$, player j would optimally accept it since, by the one-stage property, j would obtain zero payoff if he was excluded from a coalition. This contradicts the assumption of the threshold rule.

Step 1-2: We show that $\sum_{j\in T} p_j \geq v(T)$ if $T \in \mathcal{S}(N)$ includes a player i such that i is a proposer after i rejects an offer in some history. Suppose that there is a coalition $T \in \mathcal{S}(N)$ with $\sum_{j\in T} p_j < v(T)$. Suppose that a proposer offers p_i to i. Then i can deviate to rejecting it and then proposing $(T, (x_j)_{j\in T})$ where $x_i = v(T) - \sum_{j\neq i} p_j$, and $x_j = p_j$ for all $j \neq i$. This deviation is profitable since $x_i > p_i$ by the assumption, and i's offer is accepted by all members in $T \setminus \{i\}$.

Step 1-3: We show that $\sum_{j\in T} p_j \geq v(T)$ for all $T \in \mathcal{S}(N)$. Since the protocol has rich patterns, there is a player i such that i is a proposer after i rejects an offer in some history. Suppose that there is a coalition $T' \in \mathcal{S}(N)$ with $\sum_{j\in T'} p_j < v(T')$. For any $k \in T'$, $p_k <$

$v(T') - \sum_{j\in T'\setminus\{k\}} p_j$. Therefore k should make an accepted offer. By Step 1-2, player i cannot be included in T', and we have $p_i \geq v(T'\cup\{i\}) - \sum_{j\in T'} p_j > v(T'\cup\{i\}) - v(T') \geq 0$, where the last inequality follows from super-additivity of v. Again by Step 1-2, $p_k \geq v(T) - \sum_{j\in T\setminus\{k\}} p_j$ for all $T \ni i, k$. This implies that player k proposes to a coalition that does not contain player i while $p_i > 0$, which contradicts Step 1-1.

Step 1-4: We show that the equilibrium allocation belongs to the core. We have $\sum_{j\in N\setminus\{i\}} p_j = \sum_{j\in S^i\setminus\{i\}} p_j$ for every $i \in N$ who makes an accepted offer. Since i makes a best offer for himself among all accepted offers to S^i, we have $x_i^i = v(S^i) - \sum_{j\in S^i\setminus\{i\}} p_j$, and $x_j^i = p_j$ for all $j \in S^i \setminus \{i\}$. we have $\sum_{j\in N} x_j^i = x_i^i + \sum_{j\in N\setminus\{i\}} p_j = v(S^i) \leq v(N)$. By super-additivity, we have $\sum_{j\in N} x_j^i \leq v(N)$. If $\sum_{j\in N} x_j^i < v(N)$, i can earn more by proposing $(N, (x_k)_{k\in N})$ with $x_i = v(N) - \sum_{j\in N\setminus\{i\}} p_j$ and $x_j = p_j$ for $j \neq i$. Therefore $\sum_{j\in N} x_j^i = v(N)$ is satisfied, which implies that any accepted offer is efficient. As we showed $\sum_{j\in T} p_j \geq v(T)$ for all $T \in \mathcal{S}(N)$, the equilibrium outcome is a core allocation.

Step 2: Assume that the statement holds true for all (N, v) with the number of players smaller than n. Suppose that player i offers $(S^i, (x_j^i)_{j\in S^i})$ which is accepted. Let x_k^i for $k \notin S^i$ be the continuation payoffs in this subgame, and p_i be the threshold price of player i when all of n players are active. By the induction hypothesis, the continuation payoff profile $(x_k^i)_{k\in N\setminus S^i}$ is a core allocation of the restricted game consisting of remaining players. In particular we have $\sum_{k\in N\setminus S^i} x_k^i = v(N \setminus S^i)$.

We can show that $\sum_{j\in T} p_j \geq v(T)$ for all $T \in \mathcal{S}(N)$ by the same way as in Step 1. First, we show that $j \notin S^i$ implies $x_j^i \geq p_j$ if i's offer is accepted. Suppose that $j \notin S^i$ and $x_j^i < p_j$. Since the protocol has rich patterns there exists a history in which some player k ($\neq j$) makes an offer, j rejects it, and the next proposer is i. Then in the subgame after k offered to j an allocation x_j with $p_j > x_j > x_j^i$, player j would optimally accept it. This contradicts the assumption of the threshold rule.

Second, we show that $j \notin S^i$ implies $x_j^i = p_j$ if i's offer is accepted. We have $v(N \setminus S^i) = \sum_{j\in N\setminus S^i} x_j^i \geq \sum_{k\in N\setminus S^i} p_j$. On the other hand, $\sum_{j\in N\setminus S^i} p_j \geq v(N \setminus S^i)$ has been shown. Therefore we have $x_j^i = p_j$ for all $j \notin S^i$.

Third, we show that the equilibrium allocation belongs to the core. Suppose that player i's offer is accepted. Since i makes a best offer for himself among all accepted offers to S^i, we have $x_i^i = v(S^i) - \sum_{j\in S^i\setminus\{i\}} p_j$, and $x_j^i = p_j$ for all $j \in S^i \setminus \{i\}$. Then,

$$\begin{aligned}\sum_{j\in N} x_j^i &= x_i^i + \sum_{j\in S^i\setminus\{i\}} p_j + \sum_{k\in N\setminus S^i} x_k^i \\ &= v(S^i) + v(N \setminus S^i) \leq v(N),\end{aligned}$$

where the last inequality follows from super-additivity of v. If a strict inequality holds, i can earn more by proposing $(N, (x_k)_{k \in N})$ with $x_i = v(N) - \sum_{j \in N \setminus \{i\}} p_j$ and $x_j = p_j$ for $j \neq i$ since $p_j = x^i_j$ for all $j \neq i$. Therefore we must have $\sum_{j \in N} x^i_j = v(N)$, which implies that all continuation payoffs starting at a proposal node with all players active are efficient in equilibrium. As we showed $\sum_{j \in T} p_j \geq v(T)$ for all $T \in \mathcal{S}(N)$, the equilibrium outcome is a core allocation. □

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